Group Theory

Week #4, Lecture #13

I Factor groups

Prop Let $N \trianglelefteq G$ be a normal subgroup (i.e. $gxg^{-1} \in N$, $\forall x \in N$ and $g \in G \iff gN = Ng$ $\forall g \in G$). Then the left cosets of N in G form a group, under the operation

$$aN \cdot bN := abN$$

Remarks

(1) Same result holds for right cosets, with operation $Na \cdot Nb = Nab$ (right cosets $\Updownarrow$ for normal subgroups left cosets)

(2) We will call this group the factor group ($\Updownarrow$ quotient group) of G by N and denote it by G/N (verbalized as G mod N) (or, rarely, $N \backslash G$)

Proof of Prop Checks:

(0) Well defined operation (i.e., $aN \cdot bN = abN$ does not depend on the choices of a and b as coset repr.)

$$aN \cdot bN \underset{\substack{\uparrow \\ \text{by assoc of } \cdot \text{ in } G}}{=} a(Nb)N \underset{\substack{\uparrow \\ \text{since } N \trianglelefteq G}}{=} a(bN)N$$

$$\underset{\substack{\uparrow \\ \text{by assoc}}}{=} (ab)(NN) \underset{\substack{\uparrow \\ \text{since } N \leq G}}{=} ab \cdot N$$

Alternative proof (exercise): [If $aN = a'N$, $bN = b'N$, show that $ab \cdot N = a'b'N$]

(1) <u>Associativity</u> $(aN \cdot bN) \cdot cN = abNNcN$
$= ab\,NcN$
$= abc\,NN$
$= abc\,N = aN(bN \cdot cN)$ ✓

(2) <u>Identity element</u> $N = eN$

check: $(aN) \cdot N = aNN \overset{N \in G}{=} aN$
$N(aN) = NaN \underset{N \trianglelefteq G}{=} aNN = aN$

(3) <u>Inverses</u> $(aN)^{-1} = a^{-1} \cdot N$

check: $(aN)(a^{-1}N) = a(Na^{-1})N = a(a^{-1}N)N$
$= (aa^{-1})(NN) = eN = N$

similarly: $(a^{-1}N)(aN) = N$ ✓

$\boxed{\text{QED}}$

<u>Examples of factor groups</u>

(1) $G = \mathbb{Z}$, $N = n\mathbb{Z}$ $\Rightarrow$ $G/N = \mathbb{Z}/n\mathbb{Z} = \mathbb{Z}_n$

(2) $G = \mathbb{Z}_6$, $N = 2 \cdot \mathbb{Z}_6 = \{[0]_6, [2]_6, [4]_6\}$ $\binom{n=6}{m=2}$
$G/N = \mathbb{Z}_6/2\mathbb{Z}_6 = \{N, [1]_6 + N\} \cong \mathbb{Z}_2$

(3) More generally:
$$\boxed{\text{If } m|n, \text{ then } \mathbb{Z}_n/m\mathbb{Z}_n \cong \mathbb{Z}_m} \quad (*)$$
reason is based on

<u>Lemma</u> Every factor group of a cyclic group is again a cyclic group.

(This complements a previous result, which says:
every subgroup of a cyclic group is cyclic.)

<u>Proof</u> (First one — sketch) Let $G = \langle a \rangle$, and let G/N be a factor group. Then $G/N = \langle aN \rangle$ is also cyclic. ☒

(i.e.: $G = \{e, a, a^2, \ldots, a^n, \ldots\}$, then
$G/N = \{N, aN, a^2N, \ldots, a^nN, \ldots\}$) ← exercise

Back to (*): $\mathbb{Z}_n$ is cyclic, so by the lemma, $\mathbb{Z}_n / m\mathbb{Z}_n$ is also cyclic. But $|m\mathbb{Z}_n| = n/m$ and $|\mathbb{Z}_n| = n$, so $|\mathbb{Z}_n / m\mathbb{Z}_n| = \frac{n}{n/m} = m$ ☒

Projection map (from a group to a quotient group)

Prop Let G be a group, $N \trianglelefteq G$ normal subgroup. There is then a "canonical" projection map

$$\boxed{\begin{aligned} \pi : G &\longrightarrow G/N \\ a &\longmapsto aN \end{aligned}}$$

This map is a well-defined, surjective homomorphism, with kernel equal to N:

$$\boxed{\ker(\pi) = N}$$

Proof (0) Well-def: clear

(1) Homomorphism: $\pi(ab) = abN$ ✓

$\pi(a)\pi(b) = (aN)(bN) \underset{\text{assoc}}{=} a(Nb)N \underset{N \trianglelefteq G}{=} ab\,NN \underset{N \leq G}{=} abN$

(2) Surjective: clear (by definition of G/N)

(3) $\ker(\pi) = N$: $\ker(\pi) = \{a \in G : \pi(a) = N\}$ ← identity in G/N

$= \{a \in G \mid aN = N\}$ ← since $aN = bN \Leftrightarrow ab^{-1} \in N$

$= N$

Aside: GAP command for computing G/N:

FactorGroup (G, N)

Ex $G = \mathbb{Z}$, $N = n\mathbb{Z}$ $\pi : G \to G/N$ is $\pi(x) = [x]_n$

II The Fundamental Theorem of Group Homomorphisms

(a.k.a. The First Isomorphism Theorem)

Theorem Let $\varphi: G \longrightarrow G'$ be a group homomorphism, and set $K := \ker(\varphi)$. Then:

$$\boxed{G/K \cong \mathrm{im}(\varphi)}$$

isomorphic

the factor group of G by the kernel of φ (a normal subgroup of G)

the image of φ – a subgroup of G'

- In particular, if $\varphi: G \twoheadrightarrow G'$ is a surjective hom, then

$$\boxed{G/\ker(\varphi) \cong G'}$$

- More precisely, the hom. φ induces an isomorphism $\overline{\varphi}$ which fits into the commuting diagram

$$\begin{tikzcd}
G \arrow[r, "\varphi"] \arrow[d, two heads, "\pi"] & G' \\
G/K \arrow[r, dashed, "\overline{\varphi}", "\cong"'] & \varphi(G) \arrow[u, hook, "\iota"']
\end{tikzcd}$$

π: canonical proj

ι: " inclusion

$$\iota \circ \overline{\varphi} \circ \pi = \varphi$$

Proof Define $\overline{\varphi}: G/K \longrightarrow \varphi(G)$ by

$$\overline{\varphi}(aK) := \varphi(a)$$

(recall $K = \ker(\varphi)$)

(0) Well-defined: to show $\overline{\varphi}(aK) = \overline{\varphi}(bK)$ if $aK = bK$

check: $aK = bK \iff b^{-1}a \in K$ (by def of left cosets),

i.e.: $\iff \varphi(b^{-1}a) = e'$ (by def of $K = \ker(\varphi)$)

$\iff \varphi(b^{-1})\varphi(a) = e'$ (φ is a hom)

$\iff \varphi(a) = \varphi(b)$ ($\varphi(b^{-1}) = \varphi(b)^{-1}$)

$\iff \overline{\varphi}(aK) = \overline{\varphi}(bK)$ (by def of $\overline{\varphi}$)

(1) Homomorphism $\quad \bar{\varphi}(aK)\cdot\bar{\varphi}(bK) = \varphi(a)\varphi(b)$ $\quad$ as φ hom

$$= \varphi(ab)$$
$$= \bar{\varphi}(abK)$$

✓

(2) Surjectivity $\quad$ Clear $\quad$ (by def of $\varphi(G)$ and of $\bar{\varphi}$)

(3) Injectivity $\quad aK \in \ker(\bar{\varphi}) \underset{\text{def of ker}}{\Leftrightarrow} \bar{\varphi}(aK) = e' \underset{\text{def of } \bar{\varphi}}{\Longleftrightarrow} \varphi(a) = e' \underset{\text{def of } K=\ker(\varphi)}{\Longleftrightarrow} a \in K$

$\underset{\text{def of cosets}}{\Longleftrightarrow} aK = K$

$\therefore \ker(\bar{\varphi}) = \{K\}$ (the identity in G/K)

$\therefore \bar{\varphi}$ is injective

(4) Diagram commutes $\quad \iota\circ\bar{\varphi}\circ\pi\,(a) = \iota\circ\bar{\varphi}\,(aK)$

$$= \iota(\varphi(a))$$
$$= \varphi(a)$$

✓

QED

Examples

① $\mathbb{C}^\times/S^1 \cong \mathbb{R}^{>0}$ $\quad$ (exercise!)

② Recall that $GL_n(F)$ is the group of invertible $n\times n$ matrices with entries in a field F. ~~Define~~

$$\boxed{SL_n(F) = \{A \in GL_n(F) \;:\; \det A = 1\}}$$

(a) Show that $SL_n(F)$ is a normal subgroup of $GL_n(F)$

(b) Identify the factor group $GL_n(F)/SL_n(F)$.

Solution

(a) First solution: check that
- $AB^{-1} \in SL_n(F)$ $\quad \forall A,B \in SL_n F$
- $BAB^{-1} \in SL_n(F)$, $\quad \forall B \in GL_n F$, $A \in SL_n F$

Faster solution $\quad$ Identify $SL_n(F)$ as the kernel of a hom:

$$GL_n(F) \xrightarrow{\det} F^\times \qquad (\det(AB) = \det A\cdot\det B)$$
$$A \longmapsto \det A$$

Then $SL_n(F) = \ker(\det)$

and also $\det$ is surjective, since, for any $a \in F^\times$,

$$\det\begin{pmatrix} a & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix} = a \cdot 1 \cdots 1 = a$$

Hence $SL_n(F) = \ker(\det)$ is a normal subgroup.

(b)

$$\begin{tikzcd} GL_n(F) \arrow[r, two heads, "\det"] \arrow[d, "\pi"'] & F^\times \\ GL_n(F)/SL_n(F) \arrow[ur, dashed, "\widetilde{\det}", "\cong"'] & \end{tikzcd}$$

by FTH

This shows that $GL_n(F)/SL_n(F) \cong F^\times$ $\square$

eg! $GL_2(\mathbb{R})/SL_2(\mathbb{R}) \cong \mathbb{R}^\times$

$GL_2(\mathbb{Z}_3)/SL_2(\mathbb{Z}_3) \cong \mathbb{Z}_3^\times \cong \mathbb{Z}_2$

<u>Exercise</u> Let $PSL_2(F) = SL_2(F)/\left\{\pm\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\right\}$

Identify the groups $\left\{\begin{array}{l} PSL_2(\mathbb{Z}_3). \\ SL_2(\mathbb{Z}_3) \end{array}\right.$